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Making Connections between Different Weighted Measure Using Known Inequalities

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Abstract
By using well-known inequalities, we construct numerous interesting and significant links among various weighted divergence measures in this study. Basically, this work applies well-known information theory inequalities Except various relations, we tried to get bounds of $N_k^*(P;Q;W)$, $J_K^*(P;Q;W)$, $\Delta_K(P;Q;W)$, $E_k^*(P;Q;W)$, $S^*(P;Q;W)$, $L(P;Q;W)$, $\psi_M(P;Q;W)$, $R_2(P;Q;W)$ in terms of weighted divergence measures. Some relations in terms of weighted Arithmetic Mean $A(P;Q;W)$,weighted Geometric Mean $G^*(P;Q;W)$,weighted Harmonic Mean $H(P;Q;W)$,weighted Heronian Mean $N(P;Q;W)$, weighted Contra Harmonic Mean $C(P;Q;W)$, weighted Root Mean Square $S(P;Q;W)$and weighted Centroidal Mean $R(P;Q;W)$, are also obtained.

Keywords: *Standard Inequalities, weighted Divergence Measures, Convex and Normalized function, Csiszar's generalized f- Divergence Measure, Seven Standard Means.*

1 Introduction

Let

$$\Gamma_n = \{P = (p_1, p_2, \ldots, p_n), p_i > 0, \quad \sum_{i=1}^{n} p_i = 1, n \geq 2$$

be the collection of all weighted discrete probability distributions that are finite. Only for convenience, discrete distributions are limited in this case; identical findings apply to continuous distributions as well. If we take $p_i \geq 0$ for some i = 1to n ,then we have to suppose that $0\, f(0) = 0 f\left(\frac{0}{0}\right), \sum_{i=1}^{n} w_i\, p_i = 1.$

By properly specifying the function f, numerous divergence measures can be derived from these generalized f- measures. Due to its compact nature, provided by Csiszar's f- divergence, $C_f(P;Q) = \sum_{i=1}^{n} q_i\, f\left(\frac{p_i}{q_i}\right)$

And its weighted form is defined as given below by many authors as

$$C_f(P;Q;W) = \sum_{i=1}^{n} w_i\, q_i f\left(\frac{p_i}{q_i}\right), \tag{1.1}$$

where $P, Q \in \Gamma_n$.

Where f: $[0, \infty) \times (0, \infty) \rightarrow \mathcal{R}$. (Set of real nos.) is real, continuous, and convex function and $P = (p_1, p_2, \dots, p_n), p_i > 0, Q = (q_1, q_2, \dots, q_n), q_i > 0, \in \Gamma n$, where p_i and q_i are probabilities. By properly specifying the convex function f, several known divergences can be derived from these generalised measures. The following weighted divergence measurements are obtained by (1.1):

Following measure are due to (Jain and Srivastava [7]) in weighted form

$$E_k^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^{k+1}}{(p_i q_i)^{\frac{k}{2}}} \qquad k=1,3,5,7\ldots. \qquad (1.2)$$

$$J_k^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^{k+1}}{(p_i q_i)^{\frac{k}{2}}} \exp \frac{(p_i - q_i)^2}{p_i q_i} \qquad k=1,3,5,7\ldots. \qquad (1.3)$$

We are introducing the different divergences in weighted form which are well known in literature:

$$S^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i + q_i)(p_i - q_i)^2}{(p_i q_i)} \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \qquad \text{(due to[10])} \qquad (1.4)$$

$$\psi_M(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2 - q_i^2)^2}{2(p_i q_i)^{\frac{3}{2}}} \qquad \text{(due to [12])} \qquad (1.5)$$

$$L(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^2}{p_i + q_i} \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \qquad \text{(due to [11])} \qquad (1.6)$$

$R_2(P;Q;W) = \sum_{i=1}^{n} w_i \frac{p_i^2}{q_i}$ (Renyi's[13], second order entropy in weighted form)

(1.7)

$\Delta_k(P;Q;W) = \sum_{i=1}^{n} w_i \frac{|p_i - q_i|^{k+1}}{(p_i + q_i)^k}$, $k \in (0, \infty)$ (Puri and Vineze Divergence measure in weighted form due to [9] (1.8)

$$F(P;Q;W) = \sum_{i=1}^{n} w_i\, p_i \log\left(\frac{2p_i}{p_i + q_i}\right)$$

= weighted Relative Jensen- Shannon divergence (Sibson [14])

(1.9)

$$G(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i\left(\frac{þ_i+q_i}{2}\right)log\left(\frac{þ_i+q_i}{2p_i}\right)$$

= weighted Relative Arithmetic- Geometric Divergence (Taneja [15])

(1.10)

$$T(Ᵽ;Q;W)=\frac{1}{2}[G(Ᵽ;Q;W)+G(Q;Ᵽ;W)]$$

$$=\left[\sum_{i=1}^{n} w_i\left(\frac{þ_i+q_i}{2}\right)log\left(\frac{þ_i+q_i}{2\sqrt{þ_i q_i}}\right)\right], \quad (1.11)$$

= weighted Arithmetic- Geometric Mean Divergence (Taneja [15])

$$\psi(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i\frac{(þ_i+q_i)(þ_i-q_i)^2}{(þ_i q_i)} \quad (1.12)$$

= Symmetric weighted Chi – square Divergence (Dragomir &others [4])

$$J_R(Ᵽ;Q;W)=2[F(Ᵽ;Q;W)+G(Ᵽ;Q;W)]=\sum_{i=1}^{n} w_i\,(þ_i-q_i)\,log\left(\frac{þ_i+q_i}{2q_i}\right) \quad (1.13)$$

= Relative weighted J- Divergence [3]

$$h(Ᵽ;Q;W)=\frac{1}{2}\sum_{i=1}^{n} w_i\left(\sqrt{þ_i}-\sqrt{q_i}\right)^2 \quad (1.14)$$

= weighted Hellinger Discrimination [5]

$$\Delta(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i\frac{(þ_i-q_i)^2}{(þ_i+q_i)} \quad (1.15)$$

= Weighted Triangular discrimination [2]

Except above all we get below weighted divergence due to Jain and Saraswat [6]

$$N_k^*(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i\frac{(þ_i-q_i)^{2k}}{(þ_i+q_i)^{2k-1}}\exp\frac{(þ_i-q_i)^2}{(þ_i+q_i)^2}, \quad k=1,2,3\ldots \quad (1.16)$$

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II. Well Known Inequalities

The following inequalities are well-known in the literature of both pure and applied mathematics, and they are crucial for demonstrating a number of intriguing and significant findings in information theory.

$$1+\varepsilon \leq e^{\varepsilon} \leq 1+\varepsilon e^{\varepsilon} \qquad ,\ \varepsilon>0 \tag{2.1}$$

$$\frac{\varepsilon}{1+\varepsilon} \leq \log(1+\varepsilon) \leq \varepsilon \qquad ,\ \varepsilon>0 \tag{2.2}$$

III. Relations Among Various Weighted Divergence Measures

Now, using inequalities (2.1) and (2.2), we will establish bounds for some measures in terms of other weighted divergence measures as well as other significant and fascinating relationships between various divergence measures.

Proposition 1: -Let Ᵽ, Ọ $\in \Gamma_n \mathrm{x} \Gamma_n$ then we have

$$N_k^*(Ᵽ;Ọ;W) - N_{k+1}^*(Ᵽ;Ọ;W) \leq \Delta_{2k-1}(Ᵽ;Ọ;W) \tag{3.1}$$

$$\Delta_{2k+1}(Ᵽ;Ọ;W) \leq N_{k+1}^*(Ᵽ;Ọ;W) \quad , \quad k=1,2,3\ldots. \tag{3.2}$$

Proof: -Put

$$\varepsilon = \frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2} \text{ in (2.1) we get}$$

$$1+\frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2} \leq \exp\frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2} \leq 1+\frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2}\exp\frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2}$$

Now multiply by $w_i \dfrac{(ꝑ_i-ꝗ_i)^{2k}}{ꝑ_i+ꝗ_i)^{2k-1}}$, k=1,2,3…. , adding over all i = 1to n

$$\sum_{i=1}^{n} w_i \frac{(ꝑ_i-ꝗ_i)^{2k}}{(ꝑ_i+ꝗ_i)^{2k-1}} + \sum_{i=1}^{n} w_i \frac{(ꝑ_i-ꝗ_i)^{2k+2}}{(ꝑ_i+ꝗ_i)^{2k+1}} \leq \sum_{i=1}^{n} w_i \frac{(ꝑ_i-ꝗ_i)^{2k}}{(ꝑ_i+ꝗ_i)^{2k-1}} \exp\frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2}$$

$$\leq \sum_{i=1}^{n} w_i \frac{(ꝑ_i-ꝗ_i)^{2k}}{(ꝑ_i+ꝗ_i)^{2k-1}} + \sum_{i=1}^{n} w_i \frac{(ꝑ_i-ꝗ_i)^{2k+2}}{(ꝑ_i+ꝗ_i)^{2k+1}} \exp\frac{(ꝑ_i-ꝗ_i)^2}{(ꝑ_i+ꝗ_i)^2}$$

Then

$$\Delta_{2k-1}(Ᵽ;Ọ;W) + \Delta_{2k+1}(Ᵽ;Ọ;W) \leq N_k^*(Ᵽ;Ọ;W) \leq \Delta_{2k-1}(Ᵽ;Ọ;W) + N_{k+1}^*(Ᵽ;Ọ;W) \tag{3.3}$$

From 2nd & 3rd part of above we get (3.1) & from Ist & 3rd, we have (3.2).

For k=1,2,3…. We get given below (from (3.1) & (3.2))

For k= 1, $N_1^*(Ᵽ;Q;W)$ - $N_2^*(Ᵽ;Q;W) \leq \Delta_1(Ᵽ;Q;W) = \Delta(Ᵽ;Q;W)$ (3.4)

$\Delta_3(Ᵽ;Q;W) \leq N_2^*(Ᵽ;Q;W)$

For k= 2, $N_2^*(Ᵽ;Q;W)$ - $N_3^*(Ᵽ;Q;W) \leq \Delta_3(Ᵽ;Q;W)$ &

$\Delta_5(Ᵽ;Q;W) \leq N_3^*(Ᵽ;Q;W)$

For k= 3, $N_3^*(Ᵽ;Q;W)$ - $N_4^*(Ᵽ;Q;W) \leq \Delta_5(Ᵽ;Q;W)$ &

$\Delta_7(Ᵽ;Q;W) \leq N_4^*(Ᵽ;Q;W)$ and so on…

Proposition 2: -Let Ᵽ, Q $\in \Gamma_n x \Gamma_n$ then we have

$$J_k^*(Ᵽ;Q;W) - J_{k+2}^*(Ᵽ;Q;W) \leq E_k^*(Ᵽ;Q;W) \ \& \quad (3.5)$$

$$E_{k+2}^*(Ᵽ;Q;W) \leq J_{k+2}^*(Ᵽ;Q;W) \text{ where k=1,3,5,7….} \quad (3.6)$$

Proof: - Put

$\varepsilon = \frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i}$ in (2.1) we get

$$1+\frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i} \leq \exp\frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i} \leq 1+\frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i}\exp\frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i}$$

Now multiply by $w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^{k+1}}{(\mathrm{p}_i\mathrm{q}_i)^{\frac{k}{2}}}$, k=1,3,5,7…. adding over all i = 1to n

$$\sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^{k+1}}{(\mathrm{p}_i\mathrm{q}_i)^{\frac{k}{2}}} + \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^{k+3}}{(\mathrm{p}_i\mathrm{q}_i)^{\frac{k+2}{2}}} \leq \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^{k+1}}{(\mathrm{p}_i\mathrm{q}_i)^{\frac{k}{2}}} \exp\frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i}$$

$$\leq \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^{k+1}}{(\mathrm{p}_i\mathrm{q}_i)^{\frac{k}{2}}} + \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^{k+3}}{(\mathrm{p}_i\mathrm{q}_i)^{\frac{k+2}{2}}} \exp\frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\mathrm{p}_i\mathrm{q}_i}$$

Then

$$E_k^*(Ᵽ;Q;W) + E_{k+2}^*(Ᵽ;Q;W) \leq J_k^*(Ᵽ;Q;W) \leq E_k^*(Ᵽ;Q;W) + J_{k+2}^*(Ᵽ;Q;W)$$

From 2nd & 3rd part of above we get (3.5) & from Ist & 3rd, we have (3.6)

For k=1,3,5,7…. We get given below (from (3.5) & (3.6))

For k= 1, $J_1^*(Ᵽ; Q; W)$ - $J_3^*(Ᵽ; Q; W) \leq$ $E_1^*(Ᵽ; Q; W)$ &

$E_3^*(Ᵽ; Q; W) \leq$ $J_3^*(Ᵽ; Q; W)$

For k= 3, $J_3^*(Ᵽ; Q; W)$ - $J_5^*(Ᵽ; Q; W) \leq$ $E_3^*(Ᵽ; Q; W)$ &

$E_5^*(Ᵽ; Q; W) \leq$ $J_5^*(Ᵽ; Q; W)$ and so on …..

Except from Ist &2nd part of (3.7) we get

$$E_1^*(Ᵽ; Q; W) \leq J_1^*(Ᵽ; Q; W) \tag{3.7}$$

Proposition 3: -Let Ᵽ, Q $\in \Gamma_n x \Gamma_n$ then we have

$$\psi(Ᵽ; Q; W) - 2E_1^*(Ᵽ; Q; W) \leq S^*(Ᵽ; Q; W) \ \& \tag{3.8}$$

$$S^*(Ᵽ; Q; W) + \psi(Ᵽ; Q; W) \leq \psi_M(Ᵽ; Q; W) \tag{3.9}$$

Proof: - Put

$$\varepsilon = \frac{\left(\sqrt{þ_i} - \sqrt{q_i}\right)^2}{2\sqrt{þ_i q_i}} \text{ in (2.2) we get}$$

$$\frac{\frac{\left(\sqrt{þ_i}-\sqrt{q_i}\right)^2}{2\sqrt{þ_i q_i}}}{1+\frac{\left(\sqrt{þ_i}-\sqrt{q_i}\right)^2}{2\sqrt{þ_i q_i}}} \leq \log(1 + \frac{\left(\sqrt{þ_i}-\sqrt{q_i}\right)^2}{2\sqrt{þ_i q_i}}) \leq \frac{\left(\sqrt{þ_i}-\sqrt{q_i}\right)^2}{2\sqrt{þ_i q_i}} \quad \text{then}$$

$$\frac{þ_i + q_i - 2\sqrt{þ_i q_i}}{þ_i + q_i} \leq log\left(\frac{þ_i + q_i}{2\sqrt{þ_i q_i}}\right) \leq \frac{þ_i + q_i - 2\sqrt{þ_i q_i}}{2\sqrt{þ_i q_i}}$$

Multiply above by $w_i \frac{(þ_i + q_i)(þ_i - q_i)^2}{(þ_i q_i)}$, adding over all i = 1to n

$$\sum_{i=1}^{n} w_i \frac{(þ_i + q_i)(þ_i - q_i)^2}{(þ_i q_i)} \frac{þ_i + q_i - 2\sqrt{þ_i q_i}}{þ_i + q_i} \leq \sum_{i=1}^{n} w_i \frac{(þ_i + q_i)(þ_i - q_i)^2}{(þ_i q_i)} \ log\left(\frac{þ_i + q_i}{2\sqrt{þ_i q_i}}\right)$$

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$$\leq \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i+\mathrm{q}_i)(\mathrm{p}_i-\mathrm{q}_i)^2}{(\mathrm{p}_i\mathrm{q}_i)} \frac{\mathrm{p}_i+\mathrm{q}_i-2\sqrt{\mathrm{p}_i\mathrm{q}_i}}{2\sqrt{\mathrm{p}_i\mathrm{q}_i}}$$

Then

$$\sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i+\mathrm{q}_i)(\mathrm{p}_i-\mathrm{q}_i)^2}{(\mathrm{p}_i\mathrm{q}_i)} - 2\sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i-\mathrm{q}_i)^2}{\sqrt{\mathrm{p}_i\mathrm{q}_i}} \leq S^*(\mathrm{P};\mathrm{Q};W)$$

$$\leq \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i^2-\mathrm{q}_i^2)^2}{2(\mathrm{p}_i\mathrm{q}_i)^{\frac{3}{2}}} - \sum_{i=1}^{n} w_i \frac{(\mathrm{p}_i+\mathrm{q}_i)(\mathrm{p}_i-\mathrm{q}_i)^2}{(\mathrm{p}_i\mathrm{q}_i)} \text{ ,so}$$

$$\psi(\mathrm{P};\mathrm{Q};W)\text{-}2\mathrm{E}_1^*(\mathrm{P};\mathrm{Q};W) \leq S^*(\mathrm{P};\mathrm{Q};W \leq \psi_{\mathrm{M}}(\mathrm{P};\mathrm{Q};W)\text{-}\ \psi(\mathrm{P};\mathrm{Q};W) \tag{3.10}$$

From Ist & 2nd part of above we get (3.8) & from 2nd & 3rd, we have (3.9)

If we add (3.8) & (3.9) we get

$$2\,\psi(\mathrm{P};\mathrm{Q};W) \leq \psi_{\mathrm{M}}(\mathrm{P};\mathrm{Q};W) + 2\mathrm{E}_1^*(\mathrm{P};\mathrm{Q};W) \tag{3.11}$$

From 2nd & 3rd of (3.10) we have

$$S^*(\mathrm{P};\mathrm{Q};W \leq \psi_{\mathrm{M}}(\mathrm{P};\mathrm{Q};W) \tag{3.12}$$

From (3.8) & (3.12) we have

$$\psi(\mathrm{P};\mathrm{Q};W) \ \text{-}2\mathrm{E}_1^*(\mathrm{P};\mathrm{Q};W) \leq S^*(\mathrm{P};\mathrm{Q};W) \leq \psi_{\mathrm{M}}(\mathrm{P};\mathrm{Q};W) \tag{3.13}$$

Proposition 4: -Let $\mathrm{P}, \mathrm{Q} \in \Gamma_{\mathrm{n}} \mathrm{x} \Gamma_{\mathrm{n}}$ then we have

$$L(\mathrm{P};\mathrm{Q};W) + \Delta(\mathrm{P};\mathrm{Q};W) \leq \frac{1}{2}\mathrm{E}_1^*(\mathrm{P};\mathrm{Q};W) \ \& \tag{3.14}$$

$$\Delta(\mathrm{P};\mathrm{Q};W) \leq L(\mathrm{P};\mathrm{Q};W) + 2\sum_{i=1}^{n} w_i \frac{\sqrt{\mathrm{p}_i\mathrm{q}_i}(\mathrm{p}_i-\mathrm{q}_i)^2}{(\mathrm{p}_i+\mathrm{q}_i)^2} \tag{3.15}$$

Proof: - Put

$$\varepsilon = \frac{\left(\sqrt{\mathrm{p}_i}-\sqrt{\mathrm{q}_i}\right)^2}{2\sqrt{\mathrm{p}_i\mathrm{q}_i}} \text{ in (2.2) we get}$$

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$$\frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i} \le log\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right) \le \frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}$$

Multiply above by $w_i \frac{(p_i-q_i)^2}{(p_i+q_i)}$, adding over all i = 1to n

$$\sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2}{(p_i+q_i)} \frac{p_i+q_i-2\sqrt{p_iq_i}}{p_i+q_i} \le \sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2}{(p_i+q_i)} log\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right)$$

$$\le \sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2}{(p_i+q_i)} \frac{p_i+q_i-2\sqrt{p_iq_i}}{2\sqrt{p_iq_i}}$$

Then

$$\sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2}{(p_i+q_i)} - 2\sum_{i=1}^{n} w_i \frac{\sqrt{p_iq_i}(p_i-q_i)^2}{(p_i+q_i)^2} \le L(P;Q;W)$$

$$\le \frac{1}{2}\sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2}{\sqrt{p_iq_i}} - \sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2}{(p_i+q_i)}$$

Then

$$\Delta(P;Q;W) - 2\sum_{i=1}^{n} w_i \frac{\sqrt{p_iq_i}(p_i-q_i)^2}{(p_i+q_i)^2} \le L(P;Q;W) \le \frac{1}{2} E_1^*(P;Q;W) - \Delta(P;Q;W) \quad (3.16)$$

From Ist & 2nd part of above we get (3.15) & from 2nd & 3rd, we have (3.14)

Also, from (3.14) we have

$$\Delta(P;Q;W) \le \frac{1}{2} E_1^*(P;Q;W) \quad (3.17)$$

Proposition 5: -Let $P, Q \in \Gamma_n x \Gamma_n$ and $\sum_{i=1}^{n} w_i p_i = \sum_{i=1}^{n} w_i q_i = 1$,then we have

$$A(P;Q;W) \le h(P;Q;W) \le T(P;Q;W) \quad (3.18)$$

$$A(P;Q;W) + h(P;Q;W) \le \frac{1}{4}\sum_{i=1}^{n} w_i \frac{(p_i+q_i)^2}{\sqrt{p_iq_i}} \quad (3.19)$$

$$A(Ᵽ;Ọ;W)+\ T(Ᵽ;Ọ;W)\ \le \frac{1}{4}\sum_{i=1}^{n} w_i \frac{(\text{p}_i+\text{q}_i)^2}{\sqrt{\text{p}_i \text{q}_i}} \tag{3.20}$$

Here $A(Ᵽ;Ọ;W)=\sum_{i=1}^{n} w_i \left(\frac{\text{p}_i+\text{q}_i}{2}\right)$ =1= Weighted Arithmetic mean divergence

Proof: - Put

$$\varepsilon=\frac{\left(\sqrt{\text{p}_i}-\sqrt{\text{q}_i}\right)^2}{2\sqrt{\text{p}_i \text{q}_i}} \text{ in (2.2) we get}$$

$$\frac{\text{p}_i+\text{q}_i-2\sqrt{\text{p}_i \text{q}_i}}{\text{p}_i+\text{q}_i}\ \le\ log\left(\frac{\text{p}_i+\text{q}_i}{2\sqrt{\text{p}_i \text{q}_i}}\right)\le\ \frac{\text{p}_i+\text{q}_i-2\sqrt{\text{p}_i \text{q}_i}}{2\sqrt{\text{p}_i \text{q}_i}}$$

Multiply above by $w_i\left(\frac{\text{p}_i+\text{q}_i}{2}\right)$, adding over all i = 1to n

$$\sum_{i=1}^{n} w_i\left(\frac{\text{p}_i+\text{q}_i}{2}\right)\frac{\text{p}_i+\text{q}_i-2\sqrt{\text{p}_i \text{q}_i}}{\text{p}_i+\text{q}_i}\ \le\ \sum_{i=1}^{n} w_i\left(\frac{\text{p}_i+\text{q}_i}{2}\right) log\left(\frac{\text{p}_i+\text{q}_i}{2\sqrt{\text{p}_i \text{q}_i}}\right)$$

$$\le \sum_{i=1}^{n} w_i\left(\frac{\text{p}_i+\text{q}_i}{2}\right)\frac{\text{p}_i+\text{q}_i-2\sqrt{\text{p}_i \text{q}_i}}{2\sqrt{\text{p}_i \text{q}_i}}$$

$$\sum_{i=1}^{n} w_i \frac{\text{p}_i+\text{q}_i-2\sqrt{\text{p}_i \text{q}_i}}{2}\le\ T(Ᵽ;Ọ;W)\ \le\frac{1}{4}\sum_{i=1}^{n} w_i \frac{(\text{p}_i+\text{q}_i)^2}{\sqrt{\text{p}_i \text{q}_i}}-1$$

$$\frac{1}{2}\sum_{i=1}^{n} w_i\left(\sqrt{\text{p}_i}-\sqrt{\text{q}_i}\right)^2\le\ T(Ᵽ;Ọ;W)\ \le\frac{1}{4}\sum_{i=1}^{n} w_i \frac{(\text{p}_i+\text{q}_i)^2}{\sqrt{\text{p}_i \text{q}_i}}-1$$

$$h(Ᵽ;Ọ;W)\le\ T(Ᵽ;Ọ;W)\le\frac{1}{4}\sum_{i=1}^{n} w_i \frac{(\text{p}_i+\text{q}_i)^2}{\sqrt{\text{p}_i \text{q}_i}}-1 \tag{3.21}$$

From 2nd & 3rd part of above we get (3.20) & from Ist & 3rd, we have (3.19)

Except these from (3.19) & (3.21) we have

$$A(Ᵽ;Ọ;W)\le\frac{1}{4}\sum_{i=1}^{n} w_i \frac{(\text{p}_i+\text{q}_i)^2}{\sqrt{\text{p}_i \text{q}_i}} \tag{3.22}$$

$$h(\text{Ᵽ};\text{Ϙ};W) \le \frac{1}{4}\sum_{i=1}^{n} w_i \frac{(\text{ƥ}_i+\text{q}_i)^2}{\sqrt{\text{ƥ}_i\text{q}_i}} \tag{3.23}$$

& $\quad h(\text{Ᵽ};\text{Ϙ};W) \le T(\text{Ᵽ};\text{Ϙ};W)$ (3.24)

Now (3.22) - (3.23), we have

$A(\text{Ᵽ};\text{Ϙ};W) - h(\text{Ᵽ};\text{Ϙ};W) \le 0$, so $A(\text{Ᵽ};\text{Ϙ};W) \le h(\text{Ᵽ};\text{Ϙ};W)$ (3.25)

From (3.24) & (3.25) we get (3.18)

Proposition 6: -Let Ᵽ, Ϙ $\in \Gamma_n x \Gamma_n$ and $\sum_{i=1}^{n} \text{w}_i\, \text{ƥ}_i = \sum_{i=1}^{n} \text{w}_i\, \text{q}_i = 1$,then we have

$$G(\text{Ϙ};\text{Ᵽ};W) \ge \frac{1}{2} - \log 2 \tag{3.26}$$

$$\log 2 + G(\text{Ϙ};\text{Ᵽ};W) \le \frac{1}{2}[R_2(\text{Ᵽ};\text{Ϙ};W) + 1] \tag{3.27}$$

Proof: - Put

$$\varepsilon = \frac{\text{ƥ}_i}{\text{q}_i} \text{ in (2.2) we get}$$

$$\frac{\text{ƥ}_i}{\text{ƥ}_i+\text{q}_i} \le log\left(\frac{\text{ƥ}_i+\text{q}_i}{\text{q}_i}\right) \le \frac{\text{ƥ}_i}{\text{q}_i}$$

Multiply above by $w_i\left(\frac{\text{ƥ}_i+\text{q}_i}{2}\right)$, adding over all i = 1to n

$$\sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i+\text{q}_i}{2}\right)\frac{\text{ƥ}_i}{\text{ƥ}_i+\text{q}_i} \le \sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i+\text{q}_i}{2}\right) log\left(\frac{2(\text{ƥ}_i+\text{q}_i)}{2\text{q}_i}\right) \le \sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i+\text{q}_i}{2}\right)\frac{\text{ƥ}_i}{\text{q}_i}$$

$$\sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i}{2}\right) \le \log 2\sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i+\text{q}_i}{2}\right) + \sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i+\text{q}_i}{2}\right) log\left(\frac{(\text{ƥ}_i+\text{q}_i)}{2\text{q}_i}\right)$$

$$\le \sum_{i=1}^{n} w_i \frac{\text{ƥ}_i^2}{2\text{q}_i} + \sum_{i=1}^{n} w_i\left(\frac{\text{ƥ}_i}{2}\right) \text{, then}$$

$$\frac{1}{2} \le \log 2 + G(\text{Ϙ};\text{Ᵽ};W) \le \frac{1}{2}[R_2(\text{Ᵽ};\text{Ϙ};W) + 1] \tag{3.28}$$

From Ist & 2nd part of above we get (3.26) & from 2nd & 3rd, we have (3.27)

Proposition 7: -Let Ᵽ, Ϙ $\in \Gamma_n x \Gamma_n$ and $\sum_{i=1}^{n} \text{w}_i\, \text{ƥ}_i = \sum_{i=1}^{n} \text{w}_i\, \text{q}_i = 1$,then we have

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$\log 2$ - $F(Ᵽ; Q; W) \;\leq A(Ᵽ; Q; W)$ (3.29)

$\frac{1}{2} H(Ᵽ; Q; W) \;+\; F(Ᵽ; Q; W) \leq \;\log 2$ (3.30)

Here $A(Ᵽ; Q; W) = \sum_{i=1}^{n} w_i \left(\frac{\text{þ}_i + \text{q}_i}{2}\right)$ =1= Weighted Arithmetic mean divergence

& $H(Ᵽ; Q; W) \;= \sum_{i=1}^{n} 2\, \frac{w_i \text{þ}_i \text{q}_i}{p_i + q_i}$ = Weighted Harmonic mean divergence

Proof: - Put

$\varepsilon = \frac{\text{þ}_i}{\text{q}_i}$ in (2.2) we get

$$\frac{\text{þ}_i}{\text{þ}_i + \text{q}_i} \;\leq\; log\left(\frac{\text{þ}_i + \text{q}_i}{\text{q}_i}\right) \leq \frac{\text{þ}_i}{\text{q}_i}$$

Multiply above by $2 w_i \text{q}_i$, adding over all i = 1to n

$$\sum_{i=1}^{n} 2\, w_i \text{q}_i \frac{\text{þ}_i}{\text{þ}_i + \text{q}_i} \leq \sum_{i=1}^{n} 2\, w_i \text{q}_i log\left(\frac{2(\text{þ}_i + \text{q}_{i)}}{2\text{q}_i}\right) \leq \sum_{i=1}^{n} 2\, w_i \text{q}_i \frac{\text{þ}_i}{\text{q}_i} \text{ ,then}$$

$$H(Ᵽ; Q; W) \;\leq 2\;\log 2 \sum_{i=1}^{n} w_i \text{q}_i \text{- } 2\;\sum_{i=1}^{n} w_i\;\text{q}_i log\left(\frac{2\text{q}_i}{\text{þ}_i + \text{q}_i}\right) \leq 2 \sum_{i=1}^{n} w_i \text{þ}_i$$

$$H(Ᵽ; Q; W) \;\leq 2\;\log 2 \text{ - } 2\, F(Q; Ᵽ; W) \;\leq 2$$

After changing Ᵽ& Q, we get

$$H(Ᵽ; Q; W) \;\leq 2\;\log 2 \text{ - } 2\, F(Ᵽ; Q; W) \;\leq 2 \qquad (3.31)$$

From Ist & 2nd part of above we get (3.30) & from 2nd & 3rd, we have (3.29)

Some Relations: because

$$H(Ᵽ; Q; W) \;\leq\; G^*(Ᵽ; Q; W) \leq\; N(Ᵽ; Q; W) \leq\; A(Ᵽ; Q; W)$$

$$\leq R(Ᵽ; Q; W) \;\leq S(Ᵽ; Q; W) \;\leq C(Ᵽ; Q; W) \qquad (3.32)$$

It is well known inequalities in literature.

With the aid of the aforementioned inequalities, we can now obtain a few other significant relations between distinct divergences, and these are as follows.

1. from (3.18) & (3.32) , we have

$$H(Ᵽ;Q;W) \le G^*(Ᵽ;Q;W) \le N(Ᵽ;Q;W) \le A(Ᵽ;Q;W) \le h(Ᵽ;Q;W) \le T(Ᵽ;Q;W) \quad (3.33)$$

2. from (3.29) & (3.32) , we have

$$\log 2 - F(Ᵽ;Q;W) \le A(Ᵽ;Q;W) \le R(Ᵽ;Q;W) \le S(Ᵽ;Q;W) \le C(Ᵽ;Q;W) \quad (3.34)$$

3. from (3.18) & (3.29) , we have

$$\log 2 - F(Ᵽ;Q;W) \le A(Ᵽ;Q;W) \le h(Ᵽ;Q;W) \le T(Ᵽ;Q;W) \quad (3.35)$$

4. do (3.27) -(3.29), we get

$$G(Q;Ᵽ;W) + F(Q;Ᵽ;W) \le \frac{1}{2}[R_2(Ᵽ;Q;W) + 1] - A(Ᵽ;Q;W)$$

i.e., $2A(Ᵽ;Q;W)+2[G(Q;Ᵽ;W) + F(Q;Ᵽ;W)] \le R_2(Ᵽ;Q;W) + 1$

i.e., $2A(Ᵽ;Q;W)+ J_{\mathrm{R}}(Ᵽ;Q;W) \le R_2(Ᵽ;Q;W) + 1 \quad (3.36)$

5. from (3.4), (3.7) & (3.17), we get

$$\mathrm{N}_1^*(Ᵽ;Q;W) - \mathrm{N}_2^*(Ᵽ;Q;W) \le \Delta(Ᵽ;Q;W) \le \frac{1}{2}\mathrm{E}_1^*(Ᵽ;Q;W) \le \frac{1}{2}\mathrm{J}_1^*(Ᵽ;Q;W)$$

6. from (3.4) & (3.14)

$$\mathrm{N}_1^*(Ᵽ;Q;W) - \mathrm{N}_2^*(Ᵽ;Q;W) \le \Delta(Ᵽ;Q;W) \le \frac{1}{2}\mathrm{E}_1^*(Ᵽ;Q;W) - L(Ᵽ;Q;W)$$

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